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Verification of Calculations of the Potential Flow Around Two-Dimensional Foils

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An example of verification of calculations using methods based on grid-refinement studies and Richardson extrapolation is presented. The study is performed for numerical simulations of the potential flow around two-dimensional lifting foils by one low-order and two higher-order panel methods based on Morino's perturbation potential formulation. Flows with known analytical solutions have been selected to assess the reliability of the uncertainty estimations. The grid-convergence index method is the basis for these estimations. Two options are compared: estimating the uncertainty of the solution on a specific grid and of the solution found by extrapolation to zero grid cell size. The results indicate that the uncertainty estimation in the former case is more reliable than in the latter.

Introduction

IN recent years, a considerable effort has been made to establish quality standards in computational fluid dynamics (CFD). As a part of this effort, several forums have discussed procedures for verification and validation of CFD results, like the AIAA,[1] the European Research Community on Flow Turbulence and Combustion,[2] and the ITTC Resistance Committee.[3] A verification procedure is meant to give an estimation of the error of a numerical solution, whereas validation concerns the quantification of the modeling error.

Although several verification methods are discussed in the open literature,[4] in this study we have adopted the most common one, which is based on grid-refinement studies and Richardson extrapolation. In this type of method, the error estimation is based on extrapolating the solution to grid cell size zero. However, the role of this estimated exact solution can differ. Roache[4] uses it to estimate the uncertainty of the solution on a given grid, whereas Stern et al.[5] operate with "corrected solutions" and their uncertainty.

The aim of this paper is to investigate the differences in the reliability between estimating an error bar[6] of a numerical solution in a given grid and estimating the cell-size-zero solution and its uncertainty. To that end, we have chosen the two-dimensional potential flow around a foil as the test problem, because in this type of flow it is easy to select cases, like Joukowski and Kármán–Trefftz airfoils, for which analytical solutions are available. Furthermore, we have formulations of the solution method with different theoretical orders of accuracy available.[7] To assess the influence of the level of discretization adopted in the outcome of such exercise, we have kept the grid sets of our previous study[8] with a number of panels ranging from 60 to 1000.

In a previous paper,[8] we discussed the effects of the postprocessing of the data, like numerical integration and interpolation, and the use of local and global quantities. In this paper, we have selected the lift coefficient C_l obtained by integration of the surface-pressure coefficient to illustrate the results. With the present formulation of the potential flow problem, the calculation of C_l requires the determination of the pressure coefficient at the control points using the Bernoulli equation and the numerical differentiation of the velocity potential to obtain the local velocity. We are aware that in such methods the lift coefficient can be more accurately determined by the circulation required to satisfy the Kutta condition. However, our aim is to select a flow quantity that reflects the overall performance of the method.

The paper is organized in the following way: for the sake of completeness, the next section gives a brief description of the boundary-element methods. The following sections present the verification procedure and its application. The results of the verification studies are presented and discussed in a separate section, and the final section summarizes the conclusions of this study.

Boundary Element Methods

The boundary element methods (BEM) for the incompressible potential flow problem governed by the Laplace equation are based on the perturbation potential formulation from Morino.[9] The three- and two-dimensional formulations can be found, for example, in Katz and Plotkin.[10] The low-order method, designated by LO, is the standard panel method using piecewise constant source and dipole strengths. The two higher-order BEM numerical implementations, designated by HO_1 and HO_2, were selected to satisfy consistency requirements to achieve, respectively, second and third order in the typical element size Δ for the truncation error in the local expansion of the potential. Table 1 summarizes for each method the approximations used for the source σ and dipole μ singularities, the element geometry associated with each singularity, and the obtained theoretical order p_{th} for the potential ϕ.

In the element geometry description parametric parabolas are used for the quadratic approximation and parametric cubic splines for the cubic approximation. Second- and first-order differentiation schemes are used to evaluate first and second derivatives of the singularity distributions at the collocation points. In the low-order method, as in the higher-order formulations HO_1 and HO_2, the influence coefficients are calculated analytically. The surface velocities are evaluated using a third-order differentiation scheme. A detailed description of the higher-order implementations is given in Vaz et al.[7]

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Table 1 Local errors estimation results

Variable	LO	HO_1	HO_2
Tr. error	Inconsistent	$\mathcal{O}(\Delta^2)$	$\mathcal{O}(\Delta^3)$
σ	Constant	Linear	Quadratic
σ Geo.	Linear	Linear	Quadratic
μ	Constant	Linear	Quadratic
μ Geo.	Linear	Quadratic	Cubic
p_{th} of ϕ	$1 < p_{\text{th}} < 2$	2	3

Special care has been taken in the numerical implementation of the methods to avoid contamination of the results with the round-off error. For the finest discretizations adopted in this study, a careful selection of the method of solution of the system of linear equations is crucial to achieve such a result. In this particular study a lower/upper triangular matrices solver with iterative refinement is used.

Verification Procedure

In the traditional verification methods based on grid-refinement studies, the error is assumed to satisfy a power series expansion given by

$$e_{\phi_i} = \phi_i - \phi_{\text{exact}} = \sum_{j=1}^{\infty} \alpha_j h_i^{p_j} \tag{1}$$

where ϕ_i is the numerical solution of any local or integral scalar quantity on a given grid (designated by the subscript $_i$), ϕ_{exact} is the exact solution, α_j are constants, h_i is a parameter that identifies the representative grid cell size, and p_j are exponents related to the order of accuracy of the method.

In the asymptotic range, that is, when all of the high-order terms are negligible, Eq. (1) reduces to

$$\phi_i - \phi_0 = \alpha h_i^p \tag{2}$$

where ϕ_0 is the estimate of the exact solution. Equation (2) is the basis of Richardson extrapolation.

There are three unknowns in Eq. (2): ϕ_0, α, and p, where p is the observed order of accuracy. Therefore, three grids in the asymptotic range are required to estimate ϕ_0, α, and p. In a grid-convergence study with three suitable grids, the solution is convergent if 1) $(\phi_2 - \phi_1) \times (\phi_3 - \phi_2) > 0$ and 2) $p > 0$. The first condition states that the changes in ϕ are monotonous, and the second condition implies that a finite value is obtained for the grid of cell size zero.

In the well-known grid-convergence index (GCI) method[4] the numerical uncertainty U is estimated from

$$U = F_s|\delta_{\text{RE}}| \tag{3}$$

where F_s is a safety factor and δ_{RE} is the error estimation obtained by Richardson extrapolation.

$$\delta_{\text{RE}} = \phi_i - \phi_0 = \alpha h_i^p \tag{4}$$

If more than three grids are available, it is our experience[11] that the use of different grid triplets can have a significant influence on the estimated δ_{RE}. Therefore, we can alternatively compute ϕ_0, α, and p using a least-squares root approach that minimizes the function:

$$S(\phi_0, \alpha_j, p_j) = \sqrt{\sum_{i=1}^{n_g} [\phi_i - (\phi_0 + \alpha h_i^p)]^2} \tag{5}$$

where n_g is the number of grids available. The minimum of Eq. (5) is found by setting the derivatives of Eq. (5) with respect to ϕ_0, p, and α equal to zero. The details on the solution of Eq. (5) are given in Eça and Hoekstra.[11]

These two alternatives, grid triplets and groups of four or more grids using the least-squares root approach, are tested and compared in the present study.

If one wants to define an error bar for a numerical solution in a given grid of typical size h_i, the condition

$$|\phi_i - \phi_{\text{exact}}| \leq U \tag{6}$$

should be satisfied with a confidence level of 95%. With 13 grids available and the knowledge of the analytical solution, we can check if Eq. (6) is satisfied for 95% of the grid triplets chosen, using the recommended value of the safety factor[4] $F_s = 1.25$.

On the other hand, if we adopt a corrected solution obtained from the extrapolated solution to grid cell size zero ϕ_0, we need to estimate an uncertainty (which might truly be called the "uncertainty of the uncertainty") that should satisfy

$$|\phi_0 - \phi_{\text{exact}}| \leq U_0 \tag{7}$$

If we define U_0 as a percentage of δ_{RE} and follow the heuristic approach proposed by Roache,[6] we obtain

$$U_0 = \left[F_s \big/ (r^p - 1)\right] |\delta_{\text{RE}}| \tag{8}$$

that should guarantee a 95% confidence level. In a grid triplet, r is just the grid-refinement ratio of the two finest grids. In the least-squares root approach, r becomes a more ambiguous quantity. However, if one looks at the information from all of the other grids as an extra information to determine p, one can still use r defined from the grid-refinement ratio of the two finest grids. In the present study, we have kept $F_s = 1.25$ to estimate the number of triplets or groups of grids that satisfy Eq. (7), when U_0 is obtained from Eq. (8).

Results

Two types of airfoils with known analytical solutions were selected: Joukowski and Kármán–Trefftz airfoils. In these two cases, the analytical solution can be obtained by conformal mapping.

The verification studies were performed for symmetric and nonsymmetric airfoils of different camber and thickness. For each airfoil, 13 grids were generated with the usual cosine distribution of the boundary nodes along the chord. The finest grid includes 1000 elements, whereas the coarsest grid has 60 panels.

The typical cell size is defined as the inverse of the number of elements, that is, $h_i = 1/N_i$, where N_i is the number of elements of the grid.

With the number of grids available, the variety of grid triplets that can be selected is rather large. Therefore, we have limited this selection by imposing the following criteria:

1) The grid-refinement ratios have to satisfy the relation

$$0.5 \leq (h_3/h_2)/(h_2/h_1) \leq 2$$

2) The grid-refinement ratio between consecutive grids must be larger than 1.1 and smaller than 2, that is,

$$h_3/h_2 > 1.1 \wedge h_2/h_1 > 1.1, \qquad h_3/h_2 < 2 \wedge h_2/h_1 < 2$$

In the least-squares approach, we have just imposed that a minimum of four grids is used and that the maximum grid-refinement ratio between the finest and coarsest grids is between 2 and 4. It is obvious that these selection criteria limit the number of grid triplets or groups of grids available from the 13 grids available for each test case. However, the selected criteria intend to disregard the cases that one would not find in a practical test case, where the number of grids available is certainly smaller than 13.

The number of grid triplets or groups of grids that satisfy these criteria will be designated by N_t. N_c will stand for the number of cases that lead to an observed order of accuracy p between 0 and 8, that is, that satisfy the conditions of monotonous convergence. (Formally, the condition of monotonous convergence is just $p > 0$. However, in the numerical determination of p we have limited the maximum value to 8 because of the finite number of digits available.) The percentage of the N_c grid triplets or groups of grids that satisfy condition (6) is designated by V_1. V_0 is the percentage of the N_c grid triplets or groups of grids that satisfy the uncertainty requirements for the extrapolated solution [Eq. (7)].

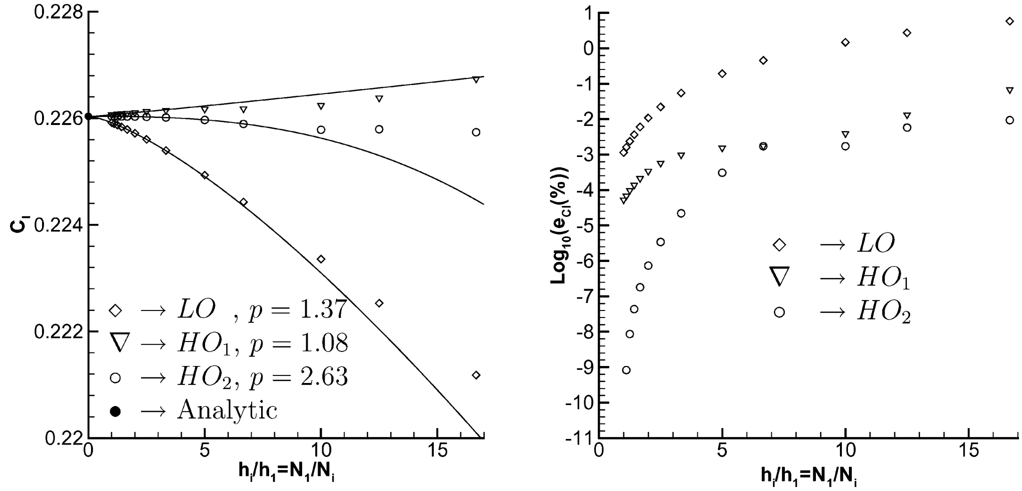

Fig. 1 ***C_l* obtained by integration of the surface-pressure coefficient as a function of the typical element size; flow around a symmetric Joukowski airfoil at 2-deg angle of attack.**

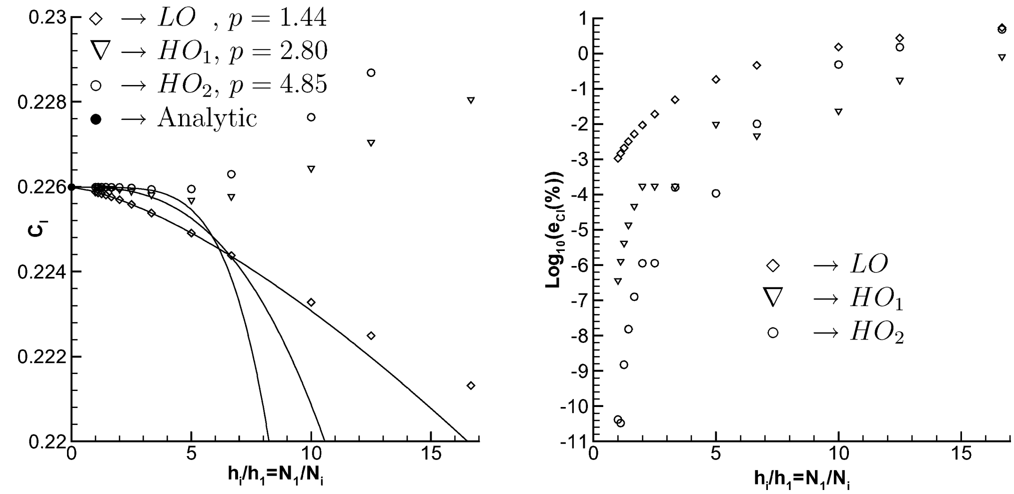

Fig. 2 ***C_l* obtained by integration of the surface-pressure coefficient as a function of the typical element size; flow around a symmetric Kármán–Trefftz airfoil at 2-deg angle of attack.**

In this study we have selected three different foils: 1) a symmetric Joukowski airfoil with thickness ratio of 4%, 2) a symmetric Kármán–Trefftz airfoil with 4% thickness ratio and a trailing-edge angle of 5 deg, and 3) a Joukowski airfoil with a camber ratio of 2% and a thickness ratio of 4%.

The flow is computed at 2-deg angle of attack for the three airfoils. As mentioned before, we have selected the lift coefficient obtained by integration of the surface-pressure coefficient C_l to illustrate the results. C_l is computed with a third-order scheme based on a cubic spline interpolation.

Figures 1–3 present C_l as a function of the typical element size for the three selected airfoils obtained with the three BEM methods. To give an idea of the dependence of the observed order of accuracy on the grid-refinement level, we have plotted the fits obtained with the least-squares root approach applied to the data of the six finest discretizations. The observed order of accuracy of the fits is indicated in the figures. Because we have the analytical solution for these three test cases, Figs. 1–3 also include the error of C_l, e_{C_l}, as a function of the typical cell size. The values of e_{C_l} are given in percentage of the analytical solution.

There are some interesting features in the data plotted in Figs. 1–3:

1) As expected, for a given grid the error decreases with the increase of the order of the method.

2) Although the plots of C_l as a function of the typical cell size suggest that there is no scatter in the data, the error e_{C_l} shows that the convergence is not perfectly smooth, especially for the coarsest grids.

3) The fits are in excellent agreement with the data of the six finest grids. However, we can observe different trends in the influence of the grid-refinement level on the convergence behavior depending on the test case and method selected:

a) There are three cases that exhibit a small influence of the typical cell size on the observed order of accuracy: the LO method in the two symmetric airfoils and the HO_1 method in the cambered Joukowski airfoil.

b) The HO_2 method for the two Joukowski airfoils shows an almost constant value of p for the finest grids. However, there is a strong dependence of p on the typical cell size for the coarsest grids.

c) The HO_1 method for the two symmetric airfoils and the HO_2 method for the symmetric Kármán–Trefftz airfoil yield values of p that do not exhibit any systematic variation with the grid-refinement level.

d) The most difficult case is the LO method for the cambered Joukowski airfoil. The results do not seem to converge to the analytical solution. However, as we have demonstrated in Eça et al.,[8] this is a consequence of the nonmonotonous behavior of the lift coefficient with the grid refinement. If we increase the number of panels, the numerical solution tends to the analytical solution. Nevertheless, the unusual behavior of the data is accompanied by a

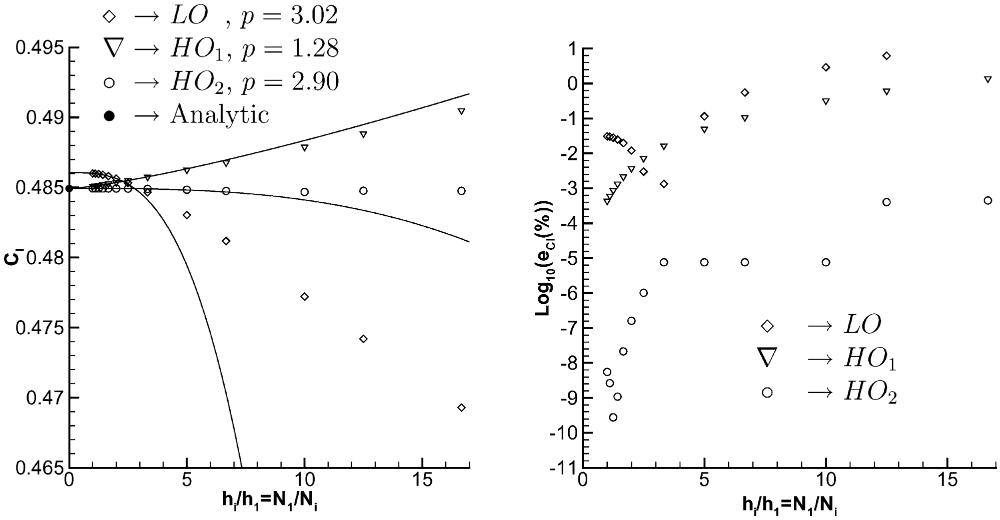

Fig. 3 ***C_l* obtained by integration of the surface-pressure coefficient as a function of the typical element size; flow around a nonsymmetric Joukowski airfoil at 2-deg angle of attack.**

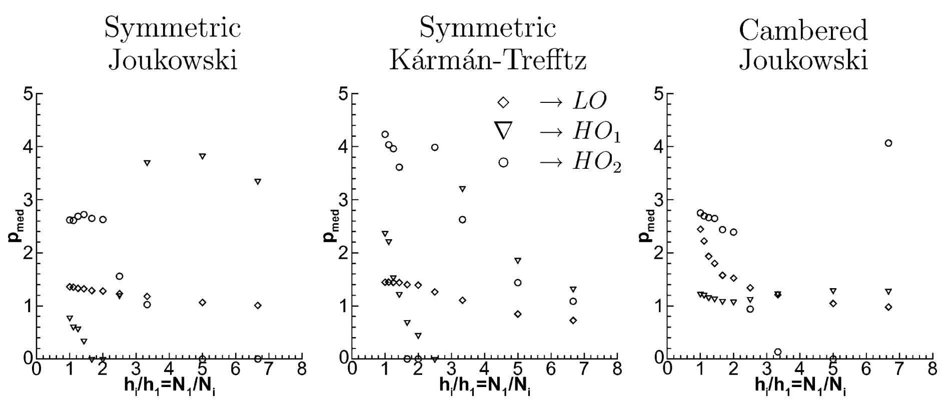

Fig. 4 Mean observed order of accuracy as a function of the typical element size: least-squares root approach.

Table 2 Results of the uncertainty estimation using grid triplets in the three test cases ($N_t = 66$)

	Symmetric Joukowski			Symmetric Kármán–Trefftz			Nonsymmetric Joukowski		
Method	N_c	V_1	V_0	N_c	V_1	V_0	N_c	V_1	V_0
LO	66	100	100	66	100	100	66	24.2	19.7
HO_1	56	76.8	85.7	61	95.1	77.0	66	97.0	100
HO_2	60	100	66.7	58	98.3	41.4	58	86.2	46.6

Table 3 Results of the uncertainty estimation using the least-squares root approach in the three test cases ($N_t = 101$)

	Symmetric Joukowski			Symmetric Kármán–Trefftz			Nonsymmetric Joukowski		
Method	N_c	V_1	V_0	N_c	V_1	V_0	N_c	V_1	V_0
LO	101	100	100	101	100	100	101	16.8	24.8
HO_1	71	83.1	84.5	87	95.4	88.5	101	100	100
HO_2	93	100	80.7	71	94.4	42.3	94	98.9	53.2

strange increase of the observed order of accuracy for the finest grids.

The trends just discussed are illustrated in Fig. 4, which presents the mean value of the observed order of accuracy p_{med} as a function of the typical cell size. p_{med} is obtained with the least-squares root approach applied to all of the groups of grids with the same finest grid that satisfies the convergence conditions. A value of p_{med} equal to 0 means that none of the groups of grids satisfies the convergence conditions.

The present set of data cover a wide range of situations where we can check the performance of the GCI in the estimation of the numerical uncertainty for a given solution or for the solution extrapolated to cell size zero. Therefore, Tables 2 and 3 presents N_c, V_1, and V_0 obtained with grid triplets and the least-squares root approach for the three test cases.

One of the important features of the results presented in Tables 2 and 3 is that N_t is not equal to N_c for five of the nine grid-refinement studies performed and in the worst case the value of N_c is only 70% of N_t. This means that the success of the application of an error estimator based on Richardson extrapolation is clearly dependent on the data behavior. Nevertheless, the percentage of cases with apparent monotonous convergence is still sufficient to evaluate the performance of the GCI.

There are several cases that do not satisfy the 95% confidence level. But it is evident from Tables 2 and 3 that banding the error of the finest grid solution is more reliable than doing the same for the extrapolated cell-size-zero solution.

To have a better understanding of the reasons that lead to these results, we have included in Tables 4–10 all of the cases that do not satisfy the conditions (6) or (7). In these tables, N_{ti} is the number of grid triplets or groups of grids that have the finest grid with N_i elements, N_{ci} indicates the number of cases that produce $0 < p < 8$ and N_{f1} and N_{f0} are the number of times that Eqs. (6) or (7) are not satisfied, and $p1_{\text{med}}$ and $p0_{\text{med}}$ are the mean value of the observed order of accuracy of the N_{f1} and N_{f0} grid triplets or groups of grids.

Table 4 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a symmetric Joukowski airfoil: HO_1 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	4	2.27	1	3.58	31	31	0	——	0	——
800	10	9	1	1.78	0	——	19	9	0	——	0	——
400	3	1	0	——	0	——	4	3	3	1.20	2	1.27
300	2	1	1	0.90	0	——	4	4	4	3.70	4	3.70
200	4	4	4	4.78	4	4.78	4	4	4	3.83	4	3.83
150	3	3	3	4.05	3	4.05	1	1	1	3.35	1	3.35

Table 5 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a symmetric Joukowski airfoil: HO_2 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	0	——	8	2.63	31	31	0	——	10	2.62
900	12	12	0	——	5	2.60	16	16	0	——	4	2.61
800	10	10	0	——	4	2.63	19	19	0	——	2	2.69
700	7	7	0	——	2	2.67	9	9	0	——	0	——
600	5	5	0	——	0	0.00	9	9	0	——	1	2.59
400	3	3	0	——	1	1.67	4	4	0	——	1	1.30

Table 6 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a symmetric Kármán–Trefftz airfoil: HO_1 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	0	——	5	2.48	31	31	0	——	4	2.22
900	12	12	0	——	2	2.31	16	16	0	——	1	2.07
800	10	10	0	——	1	1.95	19	19	0	——	0	——
300	2	0	0	——	0	——	4	3	3	3.21	3	3.21
200	4	4	3	3.60	4	3.16	4	4	1	2.42	2	2.06
150	3	3	0	——	2	1.47	1	1	0	——	0	——

Table 7 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a symmetric Kármán–Trefftz airfoil: HO_2 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	0	——	12	4.75	31	31	0	——	19	4.18
900	12	12	0	——	8	4.61	16	16	0	——	9	4.00
800	10	10	0	——	6	4.30	19	9	0	——	4	3.89
700	7	7	0	——	3	4.01	9	3	0	——	1	3.64
600	5	3	0	——	1	3.50	9	0	0	——	0	——
400	3	0	0	——	0	——	4	3	3	3.99	3	3.99
300	2	1	1	7.73	1	7.73	4	4	1	3.71	4	2.62
200	4	4	0	——	3	1.90	4	4	0	——	1	1.33

All of the tables include the results obtained with grid triplets and with the least-squares root approach applied to groups of at least four grids. In general, the two approaches lead to similar results, but in some cases the least-squares root method has some significant advantages. We will highlight these cases in the discussion of the results obtained in the present nine grid-refinement studies.

Uncertainty Estimation for the Finest Grid Solution

With the safety factor of 1.25, the GCI fails to satisfy the 95% confidence level for the uncertainty of the solution on a given grid for two (least squares) or three (grid triplets) cases: the HO_1 method for the symmetric Joukowski airfoil and the LO (and HO_2) method for the nonsymmetric Joukowski airfoil.

There are two effects that can contribute to this failure: the factor of safety is too small, or the observed order of accuracy is larger than the theoretical order of the method. These two possibilities have been explored for these two test cases:

1) In the symmetric Joukowski airfoil, the results of Tables 4 and 5 show that most of the cases that do not bound the error exhibit observed orders of accuracy much larger than the one obtained from the fit to the six finest grids. In this case, the safety factor required to satisfy the 95% confidence level would be close to 100. On the other hand, if one disregards all of the cases with an observed order of accuracy larger than the theoretical order of the method, the safety factor of 1.25 is sufficient to satisfy the 95% target. However, the number of groups of grids that are retained drops to 62 instead of the 71 included in Tables 2 and 3 for the least-squares root approach. These results indicate that the GCI performance deteriorates when the observed order of accuracy is larger than the theoretical order of the method. In this case, the use of the least-squares approach is clearly advantageous when the finest grid data are used.

2) The calculation for the nonsymmetric Joukowski airfoil with the LO method is the most troublesome of all. The failure of the GCI in this case is not unexpected because the convergence is not

Table 8 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for nonsymmetric Joukowski airfoil: LO method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	16	3.52	16	3.52	31	31	31	2.44	31	2.44
900	12	12	12	2.91	12	2.91	16	16	16	2.22	16	2.22
800	10	10	10	2.40	10	2.40	19	19	19	1.93	17	1.96
700	7	7	7	2.14	7	2.14	9	9	9	1.80	6	1.84
600	5	5	5	1.88	5	1.88	9	9	9	1.58	4	1.57
500	3	3	0	——	1	1.62	4	4	0	——	1	1.50
400	3	3	0	——	2	1.39	4	4	0	——	1	1.31

Table 9 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a nonsymmetric Joukowski airfoil: HO_1 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
200	4	4	1	1.36	0	——	4	4	0	——	0	——
150	3	3	1	1.41	0	——	1	1	0	——	0	——

Table 10 Grid triplets or groups of grids for which the uncertainty estimates do not bound the error for a nonsymmetric Joukowski airfoil: HO_2 method

	Grid triplets						Least-squares root					
N_i	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$	N_{ti}	N_{ci}	N_{f1}	$p1_{med}$	N_{f0}	$p0_{med}$
1000	16	16	8	3.17	12	2.90	31	31	0	——	19	2.74
900	12	12	0	——	8	2.78	16	16	0	——	9	2.69
800	10	10	0	——	7	2.71	19	19	0	——	10	2.66
700	7	7	0	——	2	2.64	9	9	0	——	1	2.65
600	5	5	0	——	1	2.65	9	9	0	——	3	2.33
500	3	3	0	——	0	——	4	4	0	——	1	2.32
400	3	3	0	——	1	2.24	4	4	0	——	0	——
150	3	0	0	——	0	——	1	1	1	4.07	1	4.07

monotonous. However, in the range of panels applied in this exercise, the monotonous convergence conditions are satisfied by all of the grid triplets and groups of grids tested. This requires once more a careful application of the GCI.

In this case, the safety factor required to satisfy the 95% target for the data that satisfy the selection criteria are close to 20, which is once more an unacceptably large value. On the other hand, if we consider only the groups of grids that exhibit an observed order of accuracy smaller than 2, which is the maximum value of the theoretical order of the method, a factor of safety of 3.75 is sufficient to attain the 95% target. However, in this case, only 46 of the 101 groups of grids tested are selected.

The trend observed in the data included in Tables 8–10 with a systematic growth of the observed order of accuracy with the grid refinement is typical of the convergence obtained in this test case. In these conditions, the GCI only works properly for the grid densities that are not close to the maximum or minimum exhibited by the solution, which in this case are the discretizations using fewer than 500 elements, that fortunately are in the range of practical calculations. However, in most practical applications the analytical solution is not known, and so the identification of this type of problems will require inevitably more than three grids.

For the HO_2 method in the calculation of the flow around the nonsymmetric Joukowski airfoil, the use of the least-squares root approach performs better than the grid triplets when the finest grid data are included. For the coarsest grids data, the least-squares root approach leads to larger values of N_{ci}/N_{ti} than the grid triplets in several cases.

Uncertainty Estimation for the Solution Extrapolated to Grid Cell Size Zero

The 95% confidence level is not satisfied in six out of nine test cases. The three cases where the estimation of U_0 works properly are the two symmetric airfoils with the LO method and the nonsymmetric Joukowski airfoil with the HO_1 method. These three test cases are the ones where the observed order of accuracy is less dependent on the grid-refinement level, as the agreement between the fits plotted in Figs. 1–3 and the data of the coarsest grids illustrates.

In several test cases, leaving aside the LO method in the nonsymmetric Joukowski airfoil, the number of cases that bound the error is significantly smaller than 95%. Furthermore, the results of Tables 4–10 show that the failure to bound the error occurs for almost all of the levels of grid refinement available.

It is obvious that the target of 95% confidence level can be reached by increasing the safety factor and/or neglecting the cases where the observed of accuracy is larger than the theoretical order. However, the safety factors required to attain the desired result are larger for the extrapolated solution to cell size zero than for the finest grid solution. Therefore, the present results indicate that a reliable uncertainty estimate for the solution extrapolated to grid cell size zero requires a very well-behaved observed order of accuracy, that is, an observed order of accuracy that does not change much with the grid-refinement level.

Conclusions

This paper presents a verification study for panel methods of different orders of accuracy applied to the calculation of the potential flow around Joukowski and Kármán–Trefftz airfoils with known analytical solutions. The knowledge of the analytical solution is fundamental to assess the performance of the grid-convergence index method in the estimation of the numerical uncertainties.

The present study focuses on the differences between the estimation of the uncertainty of a numerical solution on a given grid and the extrapolation of a solution to grid cell size zero with its associated uncertainty.

The calculation of the flow around two symmetric airfoils and one nonsymmetric airfoil suggests the following conclusions:

1) It is much harder to estimate the uncertainty of the solution extrapolated to grid cell size zero than to estimate the uncertainty of a numerical solution on a given grid. A reliable estimate for solutions extrapolated to grid cell size zero requires a smoothness of the convergence behavior that is nearly impossible to obtain in practical calculations.

2) The grid-convergence index method with a safety factor of 1.25 has proved to be a reliable method as long as the observed order of accuracy is not significantly larger than the theoretical order of the method. Otherwise, the method can fail to meet the 95% confidence level, and it might not be enough as a remedy to increase the factor of safety.

3) The least-squares root method is more robust than a procedure using grid triplets. The implication is that reliable uncertainty estimates might require solutions on more than three grids.

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